J. Fluid Mech. (1978), *vol.* 87, *part* 2, *pp.* 395–399
Printed in Great Britain

REVIEW

Linear and Nonlinear Waves. By G. B. WHITHAM. Wiley-Interscience, 1974. 636 pp. $25.75

This is an important book. Having recorded that opinion, I might well have completed this review by reprinting Whitham's introductory chapter, which provides an admirable summary of the splendid things to follow. Both custom and the copyright laws deny me this logical procedure, but it is not inappropriate to preface my own summary with a few of the author's preliminary statements.

> Wave motion is one of the broadest scientific subjects and unusual in that it can be studied at any technical level. The behavior of water waves and the propagation characteristics of light and sound are familiar from everyday experience...and almost any field of science or engineering involves some questions of wave motion. ...This book is an account of the underlying mathematical theory with emphasis on the unifying ideas and the main points that illuminate the behavior of waves. Most of the typical techniques for solving problems are presented, but these are not pursued beyond the point where they cease to give information about the nature of waves and become exercises in 'mathematical methods', difficult and intriguing as these may be...This is not the purpose of the book...
>
> The mathematical ideas are liberally interspersed with discussion of specific cases and specific physical fields...Many of these topics are related to some branch of fluid mechanics...This is unavoidable...[but] the account is not writter specifically for fluid dynamicists. The ideas are presented in general, and topics for application or motivation are chosen with the general reader in mind. It is assumed that flood waves in rivers, waves in glaciers, traffic flow, sonic booms, blast waves, ocean waves from storms, and so on, are of universal interest.

True to his word, Whitham focuses squarely on the ideas, both mathematical and physical, and he does cover an impressive variety of wave phenomena. Nevertheless, I believe that his presentation is suited primarily, even though not 'specifically', to fluid dynamicists and that the 'general reader' would find many impediments. This is not to deny the importance of the book for many other branches of physics, notably optics and plasma dynamics, but the outlook is that of fluid dynamics in the finest traditions of our profession; a specialist from some other discipline who made his way from cover to cover very likely would emerge thinking like a fluid dynamicist.

The book is divided into two parts: Part I (chapters 2–10, 330 pages) deals with *hyperbolic waves,* namely those described by hyperbolic partial differential equations. Part II (chapters 11–17, 257 pages) deals with *dispersive waves.* There is a list of 175 references, all of which presumably are cited in the text; however, the subject of nonlinear waves is progressing much too rapidly to permit, and the author does not attempt, a definitive bibliography. The presentation is self-contained in respect of the most important applications, gas dynamics and water waves, but the reader evidently is expected to have a basic understanding of linear wave theory and a fairly sophisticated command of applied mathematics (at the level of Jeffreys & Jeffreys, *Mathematical Physics,* 1950). The book was developed as a text for applied mathematics students and certainly provides the basis for a very challenging course, although the absence of exercises poses difficulties. I think it likely, however, that it will prove to be more important for the working scientist. Indeed, on first reading it, I several times found my mind, and my pencil, prompted to new problems, only to find their solutions anticipated by Whitham at some later point; this is surely scientific exposition of the highest order.

Misprints are surprisingly few and (among those that I noticed) relatively insignificant, and the only negative criticisms that I can offer, aside from the aforementioned absence of exercises, are minor: the Heaviside form of the Laplace transform is used (which reminds us that the author was a student of Sydney Goldstein), although only briefly (§10.1); the rather archaic elliptic integral D is introduced (§16.15) without an explicit definition, albeit with a reference to Jahnke & Emde (it is perhaps impressive evidence of the author's success in avoiding 'exercises in "mathematical methods"' with their appeal to special functions that he finds it unnecessary to reference Abramowitz & Stegun); *wave slowness* is not explicitly introduced (see below).

Chapter 2 deals with the first-order equation

$$\phi_t + c(\phi)\,\phi_x = 0, \tag{1}$$

which arises naturally in dealing with kinematic waves. Whitham, like Thomson & Tait (*Treatise on Natural Philosophy*, 1879) and Truesdell & Toupin (*The Classical Field Theories*, 1960) before him, appropriately places heavy emphasis on kinematics. This permits him to deal with characteristics and shock waves in a rather simpler context than that of classical gas dynamics. Moreover, it enables him to demonstrate the importance of these concepts for the subsequent treatment of dispersive waves by pointing out that (1) governs the propagation of the local wave number $\phi \equiv k$ with the group velocity $c(k) = \omega'(k)$, where ω is the local frequency. In chapter 3, the basic concepts are applied, in some detail, to traffic flow and flood waves (much of this material is due originally to Whitham and has been seminal for many subsequent developments) and, more briefly, to glaciers and exchange processes.

Chapter 4 deals with the Burgers equation

$$\phi_t + \phi\phi_x = \nu\phi_{xx}, \tag{2}$$

which describes the important effects of diffusion in real media, where shock waves necessarily have structure. It also provides, in the Cole–Hopf reduction of (2) to a linear equation through a Riccati transformation, an adumbration of the similarly motivated but far more recondite reduction of the Korteweg–deVries equation in the closing chapter.

Chapter 5 develops the theory of hyperbolic waves, primarily for systems with two independent variables and briefly for systems with more than two independent variables. Chapter 6 is a rather thorough introduction to gas dynamics, starting from the principles of conservation of mass, momentum and energy for an ideal fluid and including elementary kinetic theory and thermodynamics. The author comments that 'the reader interested only in the general development of wave theory may skim this chapter'; even so, a solid background in gas dynamics is necessary for a proper appreciation of chapters 8 and 9, which contain much of the most original material in Part I.

Chapter 7 takes up the classical wave equation,

$$\phi_{tt} = c^2\nabla^2\phi. \tag{3}$$

(As the author remarks, 'It is perhaps a novelty in a book on wave propagation to delay this so long, and to give such an extensive discussion of nonlinear effects first.') The discussion is, with one exception, limited to basic ideas, since 'the elementary aspects of interference and diffraction are well documented...and the more advanced theory rapidly becomes a matter of skilful use of "mathematical methods" rather than one of understanding the nature of waves more deeply'. The exception is geometrical optics, which provides the background for much of the author's original work (chapter 8) and to which over half of chapter 7 is devoted. Both homogeneous and anisotropic media, which go beyond (3), are discussed in this context, and the presentation is just right for the student (or the mature scientist) who has never had a thorough course in classical optics at the level of Born (*Optik*, 1933) or Sommerfeld (*Optics*, 1954). Still, I find it curious, and a minor shortcoming, that Whitham fails to introduce *wave slowness* explicitly; it does appear as the momentum-like variable $\mathbf{p}$ in the Hamiltonian theory, but it might well have been used effectively in several other places (cf. W. D. Hayes, pp. 12 ff. in *Nonlinear Waves*, edited by S. Liebovich & A. R. Seebass, Cornell, 1974).

Chapter 8, 'Shock Dynamics', extends geometrical optics to situations where 'nonlinear geometrical effects play the biggest role and the interactions with the flow behind [the shock] are not responsible for the major changes in the shock motion'. The general technique developed here, which may appropriately be described as *nonlinear geometrical optics*, is due originally to Whitham and, as he remarks, has potential applications in many contexts other than shock waves (e.g. diffraction of solitary waves; see *ZAMP* **28**, 1977, 889–902). Chapter 9, 'The Propagation of Weak Shocks', deals with the complementary problem, wherein the 'geometrical effects are accepted unchanged from linear theory [in order] to cope with more general nonlinear interactions within the wave profile'. This theory yields the surprising prediction, made independently by Kirkwood & Bethe (1942), Landau (1945) and Whitham (1950), that the strength of a weak, spherical shock decays like $r^{-1}(\log r)^{-\frac{1}{2}}$, rather than the r^{-1} predicted by acoustical theory (which is not uniformly valid as $r \uparrow \infty$), and represents perhaps the premier application of the more general coordinate-straining technique developed by Lighthill. It is applied especially and effectively to the sonic boom problem. In both of these chapters, 'intuitive ideas and approximations based on physical arguments are used to surmount the mathematical difficulties'. There is no pretence of mathematical rigour, and those perturbation procedures for which some rigour might be claimed are omitted because they are 'obvious, and...although often difficult, no longer involve new concepts about the behaviour of waves'. Here, as throughout, Whitham 'draw[s] the line at developing purely mathematical methods and stress[es], rather, approximations that are intimately related with the wave propagation'.

Chapter 10, 'Wave Hierarchies', which completes Part I, deals with those situations in which waves of different order appear simultaneously, such that the higher order waves provide a precursor of the main signal. This situation, which is familiar in linear theory, has its nonlinear counterparts in such problems as shock structure and flood waves.

Chapter 11, which opens Part II, presents the elements of linear dispersive waves and includes the basic concepts of phase velocity, group velocity, Fourier integrals, stationary phase, the Airy caustic, and the propagation of wave number and amplitude (or energy). It also includes a preliminary development, in the linear context, of the 'average variational principle', which subsequently plays such a dominant role in the nonlinear development. Chapter 12 applies these concepts to a variety of problems, notably Rayleigh's fish-line problem (capillary waves upstream, gravity waves downstream), Kelvin's ship-wave problem, capillary waves on thin sheets (as studied by G. I. Taylor), waves in stratified flows, and crystal optics (for which an elementary understanding of electromagnetic theory is presumed).

Chapter 13 develops water-wave theory, starting from the continuity and Euler equations for an incompressible, inviscid, homogeneous fluid and the assumption of irrotational flow. This last assumption rules out many waves of geophysical interest, as also does the assumption of homogeneity, but this is in keeping with the author's emphasis on basic ideas, and the required generalizations are typically straightforward (indeed, applications to both rotating and stratified flows are included in the preceding chapter, wherein the basic equations are simply posited). Luke's (1967) variational principle, which Whitham has exploited so effectively, also is introduced here and is used to re-derive both the free-surface and boundary conditions, as well as Laplace's equation for the motion within the fluid. The linear theory then is recapitulated and applied to the initial-value (Cauchy–Poisson) problem, interfacial waves, and waves on a steady stream (supplementing the treatment in chapter 12). Basic nonlinear theory comes next and covers the shallow-water, Boussinesq, Korteweg–deVries (KdV), and Stokes-expansion formulations. These developments are basic to what follows and include such important concepts as conservation laws, the balancing of nonlinear steepening by dispersive smoothing, and that hallmark of nonlinear dispersive waves, the dependence of dispersion on amplitude. Also included are cnoidal and solitary waves, a somewhat speculative generalization of the KdV equation to an integral equation that comprises full dispersion but only weak nonlinearity (I found this generalization somewhat out of keeping with the general spirit of the book), and the addition of a diffusion term to the KdV equation to study the structure of bores (this, too, is rather speculative, but the model appears to have a much greater potential generality than that in the preceding example).

Chapter 14, 'Nonlinear Dispersion and the Variational Method', begins the most original portion of the book (although every chapter bears the stamp of Whitham's originality, and only chapter 6 on gas dynamics could be regarded as conventional). The average variational principle for slowly varying waves is used to obtain the equations governing the nonlinear propagation of wave number and amplitude (or energy). The treatment is general, with the Klein–Gordon equation

$$\phi_{tt} - \phi_{xx} + V'(\phi) = 0, \tag{4}$$

where $V(\phi)$ is a potential with a nonlinear derivative $V'(\phi)$, being used as an example. The relation between the variational and multiple-scale ('two-timing') approaches is discussed, a Hamiltonian (canonical) transformation is presented, and adiabatic invariants are constructed. The development is continued in chapter 15, wherein it is shown that the modulation equations may be either hyperbolic or elliptic and that the latter imply instability. A brief treatment of nonlinear interactions through

Fourier analysis closes this chapter. The results of these two chapters then are applied, in chapter 16, to nonlinear optics and water waves. The average variational method is applied to Stokes waves, and several known results are confirmed and extended with striking economy. In particular, the aforementioned instability is invoked to provide an alternative explanation and a generalization of Benjamin's (1967) original Fourier calculation of the instability of Stokes waves. The variational method then is applied to the Korteweg–deVries equation

$$\eta_t + 6\eta\eta_x + \eta_{xxx} = 0 \tag{5}$$

to obtain an elegant and compact formulation of the modulation equations in terms of the zeros of the cubic equation that is associated with the cnoidal-wave solutions.

This brief review of chapters 14–16 does not begin to reflect the importance, even relative to the rest of the book, of Whitham's average variational principle, for an adequate appreciation of which the reader must be referred to the current literature. This method, which finds antecedents in the averaging method of Krylov & Bogoliubov (*Introduction to Nonlinear Mechanics*, 1957) and in Kuzmak's method (*J. appl. Math. Mech.* **23**, 1959, 730–744), has its limitations; in particular, it does not provide an explicit estimate of the asymptotic error that is implicit in its use (this is, of course, true for many powerful approximations, notably geometrical optics), for which reason it may ultimately give way to more systematic asymptotic procedures. Nevertheless, it has played a central role in the recent development of nonlinear wave theory, and I expect that it will remain an invaluable tool in the hands of the most able practitioners.

The grand *crescendo*, chapter 17, covers the exact solution of the Korteweg–deVries and other equations through the inverse scattering algorithm of Gardner, Greene, Kruskal and Miura (GGKM), which reduces (5) to a Gelfand–Levitan (more precisely, Marchenko) integral equation. Whitham first shows that the transformation

$$\eta = 2(\log F)_{xx}, \tag{6}$$

which is suggested by the GGKM algorithm, leads to exact solutions for interacting solitary waves (the 'discrete-spectrum' problem) more directly than does the original algorithm, although not without considerable ingenuity. He then develops the algorithm, applies it to the discrete-spectrum problem (the only problem for which the Marchenko integral equation has been solved exactly), presents Miura's transformation of the KdV equation to the modified KdV equations (in which the nonlinear term is cubic rather than quadratic), and constructs the first five, and demonstrates the existence of an infinite number of, conservation laws. Finally, he discusses the cubic Schrödinger, Sine–Gordon and Born–Infeld equations in the context of inverse scattering and, as an example of a discrete system, the Toda chain, and deals briefly with the Bäcklund transformation.

There are many books, going back over the past century, *with* which I might have compared Whitham's book in either coverage or impact, but I cannot do better than to close by comparing it *to* Rayleigh's *Theory of Sound* and by recalling that Alexander Pope 'compares Homer with... Vergil [but] compares Homer to... the Nile, pouring out his riches with a boundless overflow' (Webster's *New International Dictionary*, 1955).

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